Arithmetic on Church numerals

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Abstract

Church naturals allow us to represent numbers in pure lambda calculus. In this short note I'll explain how to define addition, multiplication, and power on Church nats. As a bonus, I'll show how to define fast growing functions.

Church repesents a natural number n as a higher order function, which I'll denote $[n]$. The function $[n]$ takes another function f and composes f with itself n times:

$$[n]\, f = \underbrace{f \circ f \cdots \circ f}_{n \text{ times}} = f^n$$

We can convert a Church nat a back to an ordinary nat by applying it to the ordinary successor function $S : \mathbb{N} \to \mathbb{N}$ given by $S\ n = n+1$: if $a = [n]$ then $a\ s\ 0$ gives us back ordinary natural number n because $a\ s\ 0$ is the n-fold application of the successor function to the number 0, which just increments it n times.

The first few Church natural numbers are:

$$\begin{aligned}
[0] &= \lambda f.\lambda z.z \\
[1] &= \lambda f.\lambda z.f z \\
[2] &= \lambda f.\lambda z.f(f z) \\
[3] &= \lambda f.\lambda z.f(f(f z))
\end{aligned}$$

Many descriptions of Church nats will view them in that way: as a function that takes *two* arguments f and z that computes $f(f(\ldots(f z)\ldots))$, but this point of view gets incredibly confusing when you try to define arithmetic on them, particularly multiplication and power. So think about $[n]f = f^n$ as performing n-fold function composition.

Successor on Church nats

Let's first define the successor function on Church nats:

$$[n+1]\, f = f^{n+1} = f \circ f^n = f \circ ([n]\, f)$$

So if a is a Church nat, then the successor is defined as

$$s\ a = \lambda f.f \circ (af) = \lambda f.\lambda z.f(afz)$$

Addition

Addition is also fairly easy:

$$[n+m]\, f = f^{n+m} = f^n \circ f^m = ([n]\, f) \circ ([m]\, f)$$

So if a, b are Church nats, then addition is defined as

$$a + b = \lambda f.(af) \circ (bf) = \lambda f.\lambda z.a(bf)$$

Multiplication

Multiplication is not much harder:

$$[n \cdot m]f = f^{n \cdot m} = (f^n)^m = [m]\ ([n]\ f)$$

So if a, b are Church nats, then multiplication is defined as

$$a \cdot b = \lambda f.a(bf)$$

Power

Power is a bit trickier:

$$[n^m]f = f^{(n^m)} = f^{\overbrace{n \cdot n \cdots n}^{m \text{ times}}} = (((f^n)^n)^n \cdots)^n = [n]\ ([n]\ (\cdots [n]\ f)) = ([m]\ [n])\ f$$

So if a, b are Church nats, then power is defined as

$$a^b = \lambda f.(a\ b)f = a\ b$$

Nice! If that explanation was confusing, here's another one. If we apply $b\ f^k$ we get $f^{b \cdot k}$, because the Church nat b composes f^k with itself b times. Therefore $b\ (b\ f^k) = f^{b^2 \cdot k}$, and so on. Therefore, $b\ (b\ \cdots (bf)) = f^{(b^a)}$. But applying the function b an a number of times, is precisely what the action of a as a Church nat is. So $(a\ b)f = f^{(a^b)}$ performs power, so $a^b = a\ b$.

Predecessor

Surprisingly, defining the predecessor on Church nats is the most difficult. I think this is due to Curry.

We define the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$:

$$f((a, b)) = (s(a), a)$$

If we start with $(0, x)$ and keep applying f we get the following sequence:

$$(0, x) \to (1, 0) \to (2, 1) \to (3, 2) \to (4, 3) \to \cdots$$

So

$$f^n((0, x))_1 = n$$
$$f^n((0, x))_2 = \begin{cases} x & \text{if } n = 0 \\ n - 1 & \text{if } n > 0 \end{cases}$$

So we can define the predecessor function:

$$p = \lambda n.n\ f\ (0, 0)$$

So that $p(0) = 0$ and $p(n) = n - 1$ for $n > 0$.

Pairs

We made use of pairs to define the predecessor, so to use pure lambda calculus we need to define pairs in terms of lambda. We represent a pair (a, b) as:

$$(a, b) = \lambda f.f \ \ a \ \ b$$

We can extract the components by passing in the function f:

$$\begin{aligned} \mathsf{fst} &= \lambda x.x \ (\lambda a.\lambda b.a) \\ \mathsf{snd} &= \lambda x.x \ (\lambda a.\lambda b.b) \end{aligned}$$

Disjoint union

Another way to define the predecessor is with disjoint unions. We take:

$$\begin{aligned} \mathsf{inl}(a) &= \lambda f.\lambda g.f\,a \\ \mathsf{inr}(a) &= \lambda f.\lambda g.ga \end{aligned}$$

Then we can define:

$$\begin{aligned} f(\mathsf{inl}(a)) &= \mathsf{inr}(a) \\ f(\mathsf{inr}(a)) &= \mathsf{inr}(s(a)) \end{aligned}$$

We can do this pattern match on an inl/inr by calling it with the two branches as arguments:

$$f(x) = x \ (\lambda a.\mathsf{inr}(a)) \ (\lambda a.\mathsf{inr}(s(a)))$$

And we can define:

$$p(n) = (n \ f \ \ \mathsf{inl}(0)) \ (\lambda x.x) \ (\lambda x.x)$$

Fast growing functions

Given any function $g : N \to N$ we can define a series of ever faster growing functions as follows:

$$\begin{aligned} f_0(n) &= g(n) \\ f_{k+1}(n) &= f_k^n(n) \end{aligned}$$

We can define this function using Church naturals:

$$f_k = k \ (\lambda f.\lambda n.nf\,n) \ g$$

If we take $g = s$ then,

$$\begin{aligned} f_0(n) &= n + 1 \\ f_1(n) &= 2n \\ f_2(n) &= 2^n \cdot n \end{aligned}$$

The function $A(n) = f_n(n)$ grows pretty quickly. We can play the same game again, by putting $g = A$, obtaining a sequence:

$$\begin{aligned} h_0(n) &= A(n) \\ h_{k+1}(n) &= h_k^n(n) \end{aligned}$$

To get a feeling for how fast this grows, consider h_1:

$$\begin{aligned} h_1(n) &= h_0^n(n) \\ &= A(A(A(\ldots A(A(n))))) \\ &= A(A(A(\ldots A(f_n(n))))) \\ &= A(A(A(\ldots f_{f_n(n)}(f_n(n))))) \end{aligned}$$

An expression like $h_3(3)$ gives us a relatively short lambda term that will normalise to a huge term. We might as well start with $g(n) = n^n$ since that's even easier to write using Church naturals:

$$\begin{aligned} g &= \lambda a.a\ a \\ A &= \lambda k.k\ (\lambda f.\lambda n.nf\,n)\ g\ k \\ h &= \lambda k.k\ (\lambda f.\lambda n.nf\,n)\ A\ k \\ 3 &= \lambda f.\lambda z.f\,(f\,(f\ z)) \\ X &= h\ 3 \end{aligned}$$

You can't write down anything close to the number X even if you were to write a hundred pages of towers of exponentials. Of course, we can continue this game, and define a sequence

$$\begin{aligned} g_0 &= \lambda a.a\ a \\ g_1 &= \lambda k.k\ (\lambda f.\lambda n.nf\,n)\ g_0\ k \\ g_2 &= \lambda k.k\ (\lambda f.\lambda n.nf\,n)\ g_1\ k \\ &\ldots \end{aligned}$$

Which can be generalised as:

$$\begin{aligned} f(g) &= \lambda k.k\ (\lambda f.\lambda n.nf\,n)\ g\ k \\ g_n &= f^n(g_0) \end{aligned}$$

So we get an even more compact, yet much larger number with:

$$\begin{aligned} f &= \lambda g.\lambda k.k\ (\lambda f.\lambda n.nf\,n)\ g\ k \\ Y &= (3\ f)\ (\lambda a.aa)\ 3 \end{aligned}$$

Of course, you can easily define much faster growing functions. But here's a challenge: what's the shortest lambda term that normalises, but takes more than the age of the universe to normalise? Or: what's the largest Church natural you can write down in less than 30 symbols?

Please let me know of any mistakes. I haven't checked for mistakes at all :)