THE PRODUCT OF GCD AND LCM

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October 9, 2022

This is the standard identity for the product of gcd and lcm:

$$\gcd(a, b) \cdot \mathrm{lcm}(a, b) = ab$$

One might wonder whether it holds that $\gcd(a, b, c) \cdot \mathrm{lcm}(a, b, c) = abc$. Unfortunately, it does not; consider $a = b = c = 2$. It does however hold that

$$\gcd(a, b, c) \cdot \mathrm{lcm}(ab, ac, bc) = abc \tag{1}$$

In fact, it also holds that

$$\gcd(ab, ac, bc) \cdot \mathrm{lcm}(a, b, c) = abc$$

To see this, think of a number as a vector of its prime factorisation:

$$2^2 \cdot 3^1 \cdot 7^2 = (2, 1, 0, 2, 0, 0, \cdots)$$

On this representation, the gcd corresponds to taking the pointwise minimum, and the lcd the pointwise maximum:

$$\begin{aligned}
\gcd((a_1, a_2, \cdots), (b_1, b_2, \cdots), (c_1, c_2, c_3, \cdots)) &= (\min(a_1, b_1, c_1), \min(a_2, b_2, c_2), \cdots) \\
\mathrm{lcm}((a_1, a_2, \cdots), (b_1, b_2, \cdots), (c_1, c_2, c_3, \cdots)) &= (\min(a_1, b_1, c_1), \max(a_2, b_2, c_2), \cdots)
\end{aligned}$$

And the product corresponds to the pointwise sum:

$$(a_1, a_2, \cdots) \cdot (b_1, b_2, \cdots) \cdot (c_1, c_2, c_3, \cdots) = (a_1 + b_1 + c_1, a_2 + b_2 + c_2, \cdots)$$

Thus, in this representation, equation (1) translates to:

$$\min(a_i, b_i, c_i) + \max(a_i + b_i, a_i + c_i, b_i + c_i) = a_i + b_i + c_i \qquad \text{(for all i)}$$

Now it is easy to see that the identity holds: fix i and assume without loss of generality that $a_i \leqslant b_i \leqslant c_i$, then the minimum reduces do a_i and the maximum to $b_i + c_i$.

We see that more generally, given n numbers instead of 3 numbers,

$$\gcd(\text{k-fold products}) \cdot \mathrm{lcm}(\text{(n-k)-fold products}) = \text{product}$$

For $n = 4$, this gives that the following values are all equal to $abcd$.

$$\begin{aligned}
&\gcd(\emptyset) \cdot \mathrm{lcm}(abcd) && (k = 0) \\
&\gcd(a, b, c, d) \cdot \mathrm{lcm}(bcd, acd, abd, abc) && (k = 1) \\
&\gcd(ab, ab, ad, bc, bd, cd) \cdot \mathrm{lcm}(ab, ab, ad, bc, bd, cd) && (k = 2) \\
&\gcd(bcd, acd, abd, abc) \cdot \mathrm{lcm}(a, b, c, d) && (k = 3) \\
&\gcd(abcd) \cdot \mathrm{lcm}(\emptyset) && (k = 4)
\end{aligned}$$

In fact, if we allow negative powers in the prime factorization, we can see that such identities hold over the positive rationals too, with gcd and lcm suitably extended.