THE SIGN OF A PERMUTATION IS MULTIPLICATIVE

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ABSTRACT. The sign of a permutation τ on $\{1 \dots n\}$ is usually defined in terms of the number of inversions, i.e. the number of $i < j$ such that $\tau(i) > \tau(j)$. The sign is then $(-1)^{\text{the number of inversions}}$. However, a permutation may be defined abstractly as a bijection $\tau : A \to A$ on a finite set A. To use the definition of sign, we must choose a total order on A, and hence the definition appears to be dependent on the chosen order. One can prove that this is not the case. The total order is shown to be a red-herring; we can generalise to any total relation (which does not require transitivity). Using this notion it is easy to see that the definition of sign does not depend on the total relation. Once this is established it immediately follows that $\operatorname{sign}(\sigma\tau) = \operatorname{sign}(\sigma)\operatorname{sign}(\tau)$.

Definition 1. We call $S \subset A \times A$ a total relation, or a complete directed graph (CDG) on A if for each $i \neq j$, either $(i, j) \in S$ or $(j, i) \in S$, but not both.

Example 2. The set $\{(i, j) : 1 \leq i < j \leq n\}$ is a CDG on $A = \{1 \dots n\}$.

Given a permutation $\tau : A \to A$ we can create a new CDG $\tau(S) := \{(\tau(i), \tau(j)) : (i, j) \in S\}$ by permuting its vertices.

Given two CDGs S and S' we can always turn S into S' by flipping the direction of some of its edges. We define

$$\operatorname{sign}(S, S') := (-1)^{\text{number of flips to turn } S \text{ into } S'}$$

Lemma 3. *Going from S to S' via S_{mid}, we have* $\operatorname{sign}(S, S') = \operatorname{sign}(S, S_{mid}) \cdot \operatorname{sign}(S_{mid}, S')$.

Proof. The sequence of flips to go from S to S' does not affect the value of the sign. If we flip an edge back and forth, this multiplies the sign by $(-1) \cdot (-1) = 1$. Hence the sign is the same, whether we go via S_{mid} or not. □

Definition 4. $\operatorname{sign}_S(\tau) := sign(S, \tau(S))$ for any CDG S.

Lemma 5. *The sign does not depend on S, so we may denote it simply* $\operatorname{sign}(\tau)$.

Proof. If we flip the direction of an edge in S then we subsequently also flip one edge in $\tau(S)$. Therefore the sign obtains two minus signs, and stays the same under flips. □

Theorem 6. $\operatorname{sign}(\sigma\tau) = \operatorname{sign}(\sigma)\operatorname{sign}(\tau)$.

Proof. Using lemma 3 and 5:

$$\begin{aligned}
\operatorname{sign}(\sigma\tau) &= \operatorname{sign}_S(\sigma\tau) \\
&= \operatorname{sign}(S, \sigma(\tau(S))) \\
&= \operatorname{sign}(S, \tau(S)) \cdot \operatorname{sign}(\tau(S), \sigma(\tau(S))) \\
&= \operatorname{sign}_S(\tau) \cdot \operatorname{sign}_{\tau(S)}(\sigma) \\
&= \operatorname{sign}(\tau) \cdot \operatorname{sign}(\sigma)
\end{aligned}$$

□