A MAGIC DETERMINANT FORMULA FOR SYMMETRIC POLYNOMIALS OF EIGENVALUES

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$$\sum_i p_i \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_n^{i_n} = \sum_i p_i \det(A_1^{i_1} | A_2^{i_2} | \cdots | A_n^{i_n})$$

Trace and determinant

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \qquad \text{(e.g. integer matrix)}$$

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$$\begin{aligned} \operatorname{tr}(A) &= \lambda_1 + \lambda_2 + \lambda_3 = A_{11} + A_{22} + A_{33} \\ \det(A) &= \lambda_1 \lambda_2 \lambda_3 = A_{11}A_{22}A_{33} - A_{11}A_{32}A_{23} - A_{12}A_{21}A_{33} + \\ &\qquad A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31} \end{aligned}$$

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Polynomial of irrational complex numbers = polynomial of integer entries!
Q: Which other polynomials $p(\lambda_1, \lambda_2, \lambda_3)$ can we compute exactly?
A: Fundamental theorem of symmetric polynomials: all symmetric ones.

$$p(\lambda_1, \lambda_2, \lambda_3) = p(\lambda_2, \lambda_1, \lambda_3) = \cdots = p(\lambda_3, \lambda_2, \lambda_1)$$

The magic method

$$p(\lambda_1, \lambda_2, \lambda_3) = \quad \lambda_1 \lambda_2^4 \lambda_3^4 \ + \ \lambda_1^4 \lambda_2 \lambda_3^4 \ + \ \lambda_1^4 \lambda_2^4 \lambda_3$$

The magic method

$$\begin{aligned} p(\lambda_1,\lambda_2,\lambda_3) &= \lambda_1 \lambda_2^4 \lambda_3^4 + \lambda_1^4 \lambda_2 \lambda_3^4 + \lambda_1^4 \lambda_2^4 \lambda_3 \\ &= \det(A_1|A_2^4|A_3^4) + \det(A_1^4|A_2|A_3^4) + \det(A_1^4|A_2^4|A_3) \end{aligned}$$

The magic method

$$\begin{aligned} p(\lambda_1, \lambda_2, \lambda_3) &= \quad \lambda_1 \, \lambda_2^4 \, \lambda_3^4 \quad + \quad \lambda_1^4 \, \lambda_2 \, \lambda_3^4 \quad + \quad \lambda_1^4 \, \lambda_2^4 \, \lambda_3 \\ &= \det(A_1 | A_2^4 | A_3^4) + \det(A_1^4 | A_2 | A_3^4) + \det(A_1^4 | A_2^4 | A_3) \end{aligned}$$

$$A_2^4 = \text{the second column of } A^4$$

THE MAGIC METHOD

$$\begin{aligned} p(\lambda_1,\lambda_2,\lambda_3) &= \lambda_1\,\lambda_2^4\,\lambda_3^4 + \lambda_1^4\,\lambda_2\,\lambda_3^4 + \lambda_1^4\,\lambda_2^4\,\lambda_3 \\ &= \det(A_1|A_2^4|A_3^4) + \det(A_1^4|A_2|A_3^4) + \det(A_1^4|A_2^4|A_3) \end{aligned}$$

$$A_2^4 = \text{the second column of } A^4$$

$$\cdots + a\lambda_1^{i_1}\cdot\lambda_2^{i_2}\cdots\lambda_n^{i_n} + \cdots$$
$$\Downarrow$$
$$\cdots + a\det(A_1^{i_1}|A_2^{i_2}|\cdots|A_n^{i_n}) + \cdots$$

The trace

$$\lambda_1 + \lambda_2 + \lambda_3$$

The trace

$$\lambda_1 + \lambda_2 + \lambda_3 = \quad \lambda_1\,\lambda_2^0\,\lambda_3^0 \;+\; \lambda_1^0\,\lambda_2\,\lambda_3^0 \;+\; \lambda_1^0\,\lambda_2^0\,\lambda_3$$

The trace

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$$= \det(A_1 | A_2^0 | A_3^0) + \det(A_1^0 | A_2 | A_3^0) + \det(A_1^0 | A_2^0 | A_3)$$

The trace

$$\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 &= \quad \lambda_1\,\lambda_2^0\,\lambda_3^0 \quad + \quad \lambda_1^0\,\lambda_2\,\lambda_3^0 \quad + \quad \lambda_1^0\,\lambda_2^0\,\lambda_3 \\
&= \det(A_1|A_2^0|A_3^0) + \det(A_1^0|A_2|A_3^0) + \det(A_1^0|A_2^0|A_3) \\
&= \det\begin{pmatrix} A_{11} & 0 & 0 \\ A_{12} & 1 & 0 \\ A_{13} & 0 & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & A_{21} & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{23} & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 0 & A_{31} \\ 0 & 1 & A_{32} \\ 0 & 0 & A_{33} \end{pmatrix}
\end{aligned}$$

The trace

$$\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 &= \quad \lambda_1\, \lambda_2^0\, \lambda_3^0 \;+\; \lambda_1^0\, \lambda_2\, \lambda_3^0 \;+\; \lambda_1^0\, \lambda_2^0\, \lambda_3 \\
&= \det(A_1 | A_2^0 | A_3^0) + \det(A_1^0 | A_2 | A_3^0) + \det(A_1^0 | A_2^0 | A_3) \\
&= \det\begin{pmatrix} A_{11} & 0 & 0 \\ A_{12} & 1 & 0 \\ A_{13} & 0 & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & A_{21} & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{23} & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 0 & A_{31} \\ 0 & 1 & A_{32} \\ 0 & 0 & A_{33} \end{pmatrix} \\
&= A_{11} + A_{22} + A_{33}
\end{aligned}$$

Prove or disprove

$$\sum_i p_i \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_n^{i_n} = \sum_i p_i \det(A_1^{i_1} | A_2^{i_2} | \cdots | A_n^{i_n})$$

Note: even if $B = S^{-1}AS$, still in general

$$\det(A_1^{i_1} | A_2^{i_2} | \cdots | A_n^{i_n}) \neq \det(B_1^{i_1} | B_2^{i_2} | \cdots | B_n^{i_n})$$

No peeking! `https://julesjacobs.com/pdf/sympoly.pdf`