Paradoxes of Probabilistic Programming

and how to condition on events of measure zero with infinitesimal probabilities (to appear at POPL'21)

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Probabilistic programming

- Domain specific language for statistical and machine learning models
- Normal programming language extended with rand, observe, and run

Probabilistic programming

Example:

- Men's height is distributed according to $\mathit{Normal}(1.8, 0.5)$ meters
- Women's height is distributed according to $\mathit{Normal}(1.7, 0.5)$ meters
- A scientist randomly samples a man and a woman and compares their height
- The scientist tells us that the heights are equal

Question: What's the expected value of the height in this situation?

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```
function meters(){
  h = rand(Normal(1.7, 0.5))
  observe(Normal(1.8, 0.5), h)
  return h
}
samples = run(meters, 1000)
estimate = average(samples)
```

Answer: ≈ 1.75

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- Men's height is distributed according to *Normal*(1.8, 0.5) meters
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```
function centimeters(){
  h = rand(Normal(170, 50))
  observe(Normal(180, 50), h)
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}
samples = run(meters, 1000)
estimate = average(samples)
```

Answer: ≈ 175

Paradox 1

Suppose the scientist is lazy, and only does the measurement half of the time...

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h = rand(Normal(1.7, 0.5))
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Centimeters:

```
h = rand(Normal(170, 50))
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Answer: ≈ 170.2

- The answer depends on whether the scientist uses meters or centimeters!

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- Happens if we run this with importance sampling in Anglican
- The issue is fundamental and not limited to Anglican
- Even happens in formal operational semantics (e.g. Commutative or Quasi-Borel)
- Unclear what the answer *should* be, or whether this program should be disallowed

Paradox 2

Objection: you shouldn't do observe a variable number of times based on coin flip

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h = rand(Normal(1.7, 0.5))
w = rand(Normal(60, 10))
if(flip(0.5)){
  observe(Normal(1.8, 0.5), h)
}else{
  observe(Normal(70, 10), w)
}
return h
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Answer: ≈ 1.75

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Logarithmic ruler program:

```
H = rand(LogNormal(1.7,0.5))
observe(LogNormal(1.8,0.5),H)
return log(H)
```

Answer: 1.62

- Whether we use linear scale or log scale shouldn't matter, just like meters or centimeters shouldn't matter

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- What do probabilistic programs really mean?
- What does probililistic conditioning really mean?
- Related to the Borel-Komolgorov paradox

Overview

Problem:

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Result:

- New language is invariant under arbitrary parameter transformations
- Programs have clear probabilistic meaning via *rejection sampling*
- Implemented as a DSL in Julia

Probabilistic programming 101: manual rejection sampling

Someone: "I rolled three dice $x, y, z \in \{1, 2, 3, 4, 5, 6\}$ and observed that $x + y = z$."

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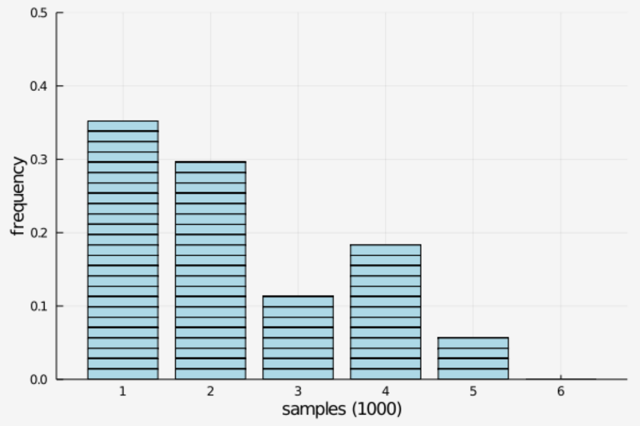

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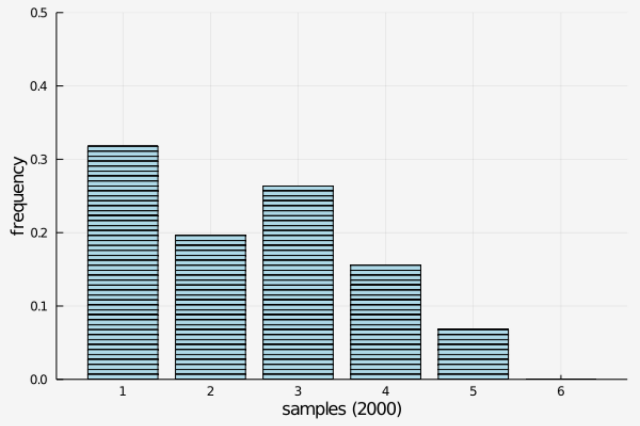

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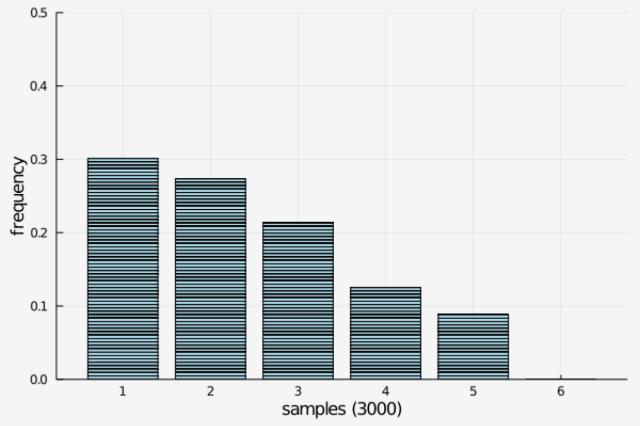

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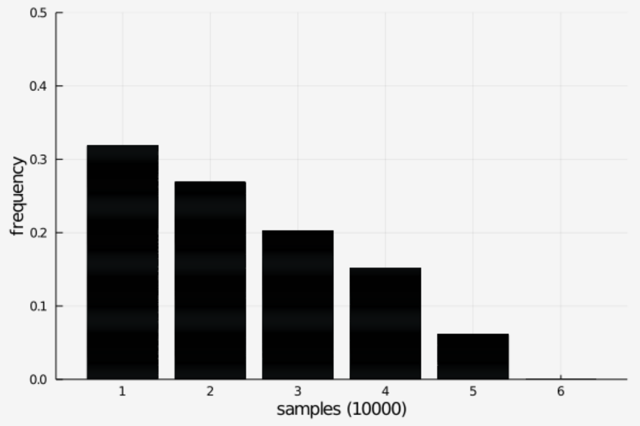

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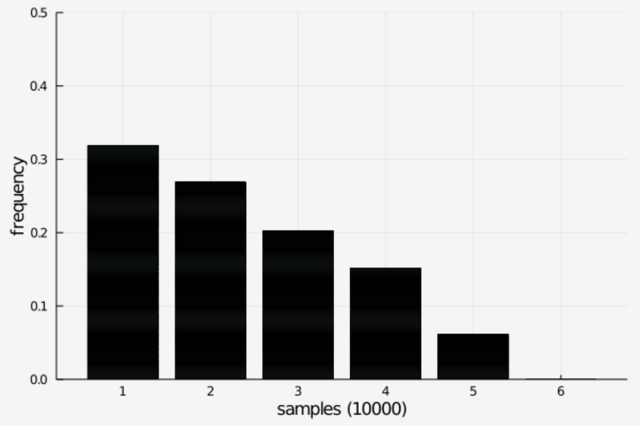

Key idea of PP: answer probabilistic inference questions by repeated simulation + filtering

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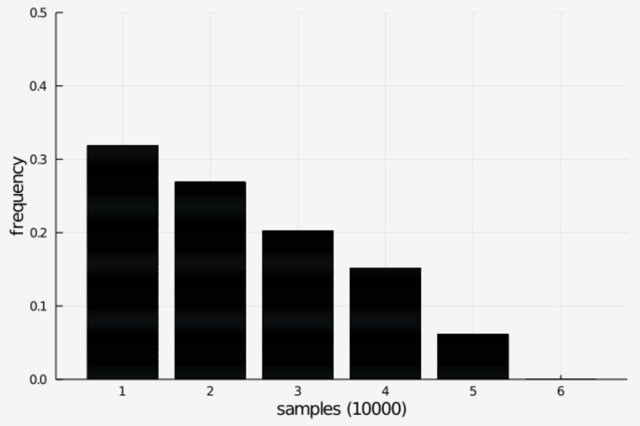

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$\Longrightarrow$ **Probabilistic Programming Language = DSL for probabilistic simulations**

Probabilistic programming 101: DSL for rejection sampling

Probabilistic programming language:

- Normal programming language + `rand(D)`
- `observe(b)` – filtering/conditioning
- `run(func, k)` – run simulation
  `func()` k times, return array of samples

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DSL implementation:

```
weight = 1
function observe(b){
  if(!b) weight = 0
}
function run(func, k){
  samples = []
  for(i in 1..k){
    weight = 1
    result = func()
    if(weight == 1){
      samples.append(result)
    }
  }
  return samples
}
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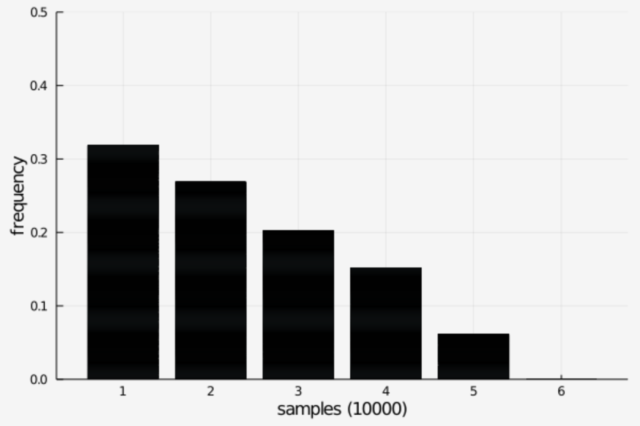

Probabilistic programming 101

```
function multiDice(){
 x = rand(DiscreteUniform(1,6))
 for(i in 1:x){
  y = rand(DiscreteUniform(1,6))
  observe(rand(DiscreteUniform(1,y)) == 3)
 }
 observe(rand(DiscreteUniform(1,6))+x == 5)
 return x
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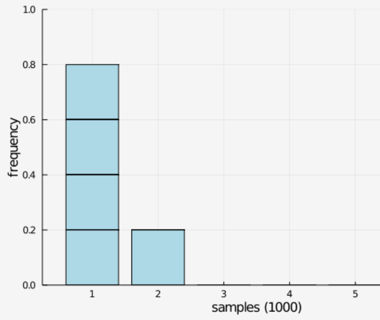

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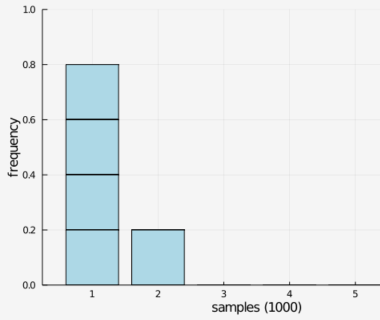

Problem: most samples get rejected $\implies$ convergence is slow

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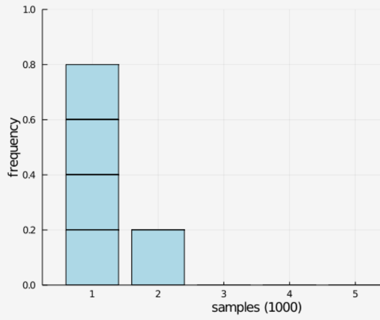

Problem: most samples get rejected $\Longrightarrow$ convergence is slow

Standard solution: importance sampling

- change `observe(rand(D) == x)` $\mapsto$ `observe(D,x)`
- `function observe(D,x){ weight *= probability(D,x) }`
- weights are now numbers between 0..1 instead of only 0,1
- `run` returns an array of weighted samples

Probabilistic programming 101: importance sampling

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 for(i in 1:x){
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 }
 observe(DiscreteUniform(1,6), 5-x)
 return x
}
samples = run(multiDice, 1000)
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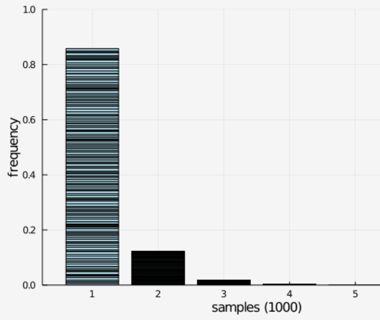

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Probabilistic programming 101: continuous distributions

Continuous distributions are problematic because `probability(D,x) = 0`.

When doing observe(D, x) for continuous distributions D,

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Intuition: pdf$(D, x) \propto$ the probability that rand(D) is close to x. **Formally:**

$$\mathrm{cdf}(D, x) = \mathbb{P}[\mathrm{rand}(D) < x] \qquad \mathrm{pdf}(D, x) = \frac{d}{dx}\mathrm{cdf}(D, x)$$

End of probabilistic programming 101. □

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End of probabilistic programming 101. $\square$

Using the pdf instead of the probability is the source of the strange behaviour!

What went wrong: conditionals

Recall the drunk scientist:

```
if(flip(0.5)){
  observe(Normal(1.8, 0.5), h)
}else{
  observe(Normal(70, 10), w)
}
```

```
function observe(D,x){
  weight *= pdf(D,x)
}
```

What went wrong: conditionals

Recall the drunk scientist:

```
if(flip(0.5)){
  observe(Normal(1.8, 0.5), h)
}else{
  observe(Normal(70, 10), w)
}
```

```
function observe(D,x){
  weight *= pdf(D,x)
}
```

- An observe(D, x) call multiplies the weight by pdf(D, x)
- The pdf is not unitless! $\text{pdf}(\text{Normal}(\mu, \sigma), x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$
- The weight has units m^{-1} in some trials and kg^{-1} in other trials
- Results in unit errors when computing the weighted average:

$$\mathbb{E}[bmi] \approx \frac{\sum_{k=1}^{N}(weight_k) \cdot (bmi_k)}{\sum_{k=1}^{N}(weight_k)}$$

- **The sum adds $m^{-1} + kg^{-1}$!**

What went wrong: nonlinear parameter transformations

Recall the log scale scientist:

`observe(Normal(1.8, 0.5), h)` vs `observe(LogNormal(1.8, 0.5), H)`

Conditioning on events of measure zero is ambiguous!

What went wrong: nonlinear parameter transformations

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Conditioning on events of measure zero is ambiguous!

$$A_\epsilon = \{(x, y) \in \mathbb{R}^2 : |x - y| \leq \epsilon\}$$
$$B_\epsilon = \{(x, y) \in \mathbb{R}^2 : |\exp(x) - \exp(y)| \leq \epsilon\}$$

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"Although the sequences A_ϵ and B_ϵ tend to the same limit "$x = y$", the conditional densities $\mathbb{P}(x|A_\epsilon)$ and $\mathbb{P}(x|B_\epsilon)$ tend to different limits. As we see from this, merely to specify "$x = y$" without any qualifications is ambiguous. Whenever we have a probability density on one space and we wish to generate from it one on a subspace of measure zero, the only safe procedure is to pass to an explicitly defined limit by a process like A_ϵ and B_ϵ. In general, the final result will and must depend on which limiting operation was specified. This is extremely counter-intuitive at first hearing; yet it becomes obvious when the reason for it is understood."
– E.T. Jaynes (paraphrased)

Solution: don't condition on measure zero events

Problem: conditioning on events of measure zero is ambiguous.
Solution: condition on intervals.

```
observe(D, Interval(x,w))
```

Meaning: rand(D) is in an interval of width w around x.

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Rejection sampling:

```
function observe(D,I){
  if(abs(rand(D) - I.midpoint) > I.width/2){ weight = 0 }
}
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Rejection sampling:

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function observe(D,I){
  if(abs(rand(D) - I.midpoint) > I.width/2){ weight = 0 }
}
```

Importance sampling:

```
function observe(D,I){ weight *= probability(D,I) }
```

For intervals, `probability(D,I)` is nonzero.

Example of conditioning on intervals

Example:

```
function centimeters(){
  h = rand(Normal(170, 50))
  if(flip(0.5)){
    observe(Normal(180, 10), Interval(h, 10))
  }
}
function meters(){
  h = rand(Normal(1.7, 0.5))
  if(flip(0.5)){
    observe(Normal(1.8, 0.1), Interval(h, 0.1))
  }
}
```

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  h = rand(Normal(170, 50))
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}
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  if(flip(0.5)){
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Same output & no unit errors, even though observe is conditionally executed!
Rejection sampling and importance sampling converge to the same answer!

Take the limit

We still want to condition on measure zero events

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Idea: parameterize the program by the width of the interval, and take the limit $\mathit{width} \to 0$

Take the limit

We still want to condition on measure zero events

Idea: parameterize the program by the width of the interval, and take the limit *width* $\to 0$

```
function drunk(width){
  h = rand(Normal(1.7, 0.5))
  w = rand(Normal(60, 10))
  if(flip(0.5)){
    observe(Normal(1.8, 0.1), Interval(h, A*width))
  }else{
    observe(Normal(70, 10), Interval(w, B*width))
  }
}
```

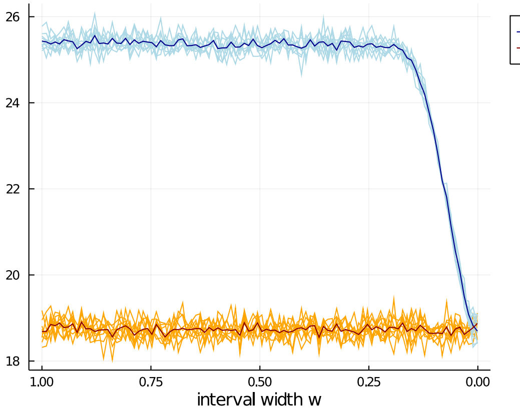
Since *width* is unitless, we must introduce constants A and B with units m and kg.
The relative size matters even as *width* $\to 0$!

Take the limit

We still want to condition on measure zero events

Idea: parameterize the program by the width of the interval, and take the limit $width \to 0$

```
function drunk(width){
  h = rand(Normal(1.7, 0.5))
  w = rand(Normal(60, 10))
  if(flip(0.5)){
    observe(Normal(1.8, 0.1), Int
  }else{
    observe(Normal(70, 10), Inter
  }
}
```



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Take the limit

We still want to condition on measure zero events

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function drunk(width){
  h = rand(Normal(1.7, 0.5))
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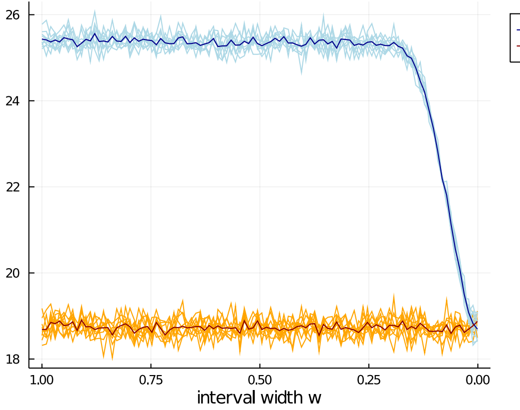

Since ***width*** **is unitless, we must introduce constants** A **and** B **with units** m **and** kg**.**
The relative size matters even as ***width*** $\to 0$**!**

Can we compute the limit $w \to 0$ **directly?**

Infinitesimal numbers

Definition

An infinitesimal number is a pair $(r, n) \in \mathbb{R} \times \mathbb{Z}$, which we write as $r\epsilon^n$.

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Arithmetic with infinitesimals:

$$r\epsilon^n \pm s\epsilon^k = \begin{cases} (r \pm s)\epsilon^n & \text{if } n = k \\ r\epsilon^n & \text{if } n < k \\ \pm s\epsilon^k & \text{if } n > k \end{cases}$$

$$(r\epsilon^n) \cdot (s\epsilon^k) = (r \cdot s)\epsilon^{n+k}$$

$$(r\epsilon^n)/(s\epsilon^k) = \begin{cases} (r/s)\epsilon^{n-k} & \text{if } s \neq 0 \\ \text{undefined} & \text{if } s = 0 \end{cases}$$

Infinitesimal numbers

The probability that rand(D) lies in the interval $[x - r\epsilon^n, x + r\epsilon^n]$:

$$\text{probability}(D, \text{Interval}(x, r\epsilon^n)) = \begin{cases} \text{cdf}(D, x + \frac{1}{2}r) - \text{cdf}(D, x - \frac{1}{2}r) & \text{if } n = 0 \\ \text{pdf}(D, x) \cdot r\epsilon^n & \text{if } n > 0 \end{cases}$$

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Infinitesimals unify cdf and pdf!

Correctness properties

Consistency with existing probabilistic programming languages:

`observe(D,Interval(x,eps))` gives the same result as `observe(D,x)` outside conditionals

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Theorem

If $f(x)$ is given by a "probability expression" and $f(\epsilon) = r\epsilon^n$, then $\lim_{x\to 0} \frac{f(x)}{x^n} = r$.

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Theorem

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Definition

We say that $f(x)$ is a "probability expression" in the variable x if $f(x)$ is defined using the operations $+, -, \cdot, /$, constants, and probability$(D, \text{Interval}(s, rx))$ where $r, s \in \mathbb{R}$ are constants, and D is a probability distribution with differentiable cdf.

Importance sampling computes a probability expression.

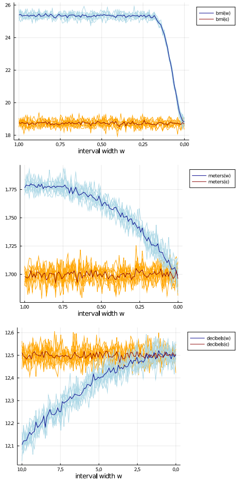
Infinitesimal numbers

Example programs:

```
function bmi(width){
  h = rand(Normal(1.70, 0.2))
  w = rand(Normal(70, 30))
  if(flip(0.5)){
        observe(Normal(2.0,0.1), Interval(h,10*width))
  }else{
        observe(Normal(90,5), Interval(w,width))
  }
  return w / h^2
}

function meters(width){
  h = rand(Normal(1.7,0.5))
  if(flip(0.5)){
        observe(Normal(2.0,0.1), Interval(h,width))
  }
  return h
}

function decibels(width){
  x = rand(Normal(10,5))
  observe(Normal(15,5),Interval(x,width))
  return x
}
```



Theorem works: we can condition on events of measure zero without paradoxes

Parameter transformations

The factor in front of ϵ allows us to do parameter transformations correctly:

$$\text{A function } f \text{ maps Interval}(x, \epsilon) \text{ to Interval}(f(x), f'(x)\epsilon).$$

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Original program:

```
h = rand(Normal(1.7,0.5))
observe(Normal(1.8,0.5),
        Interval(h,eps))
return h
```

Answer: 1.75

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```
h = rand(Normal(1.7,0.5))
observe(Normal(1.8,0.5),
        Interval(h,eps))
return h
```

Answer: 1.75

Logarithmic ruler program:

```
H = rand(LogNormal(1.7,0.5))
observe(LogNormal(1.8,0.5),
        Interval(H,H*eps))
return log(H)
```

Answer: 1.75

Same output $\Longrightarrow$ parameter transformation correctly applied

Parameter transformations

Language support for parameter transformations $f : \mathbb{R} \to \mathbb{R}$.

- Define $f(D)$ for distributions by defining rand, pdf, cdf of $f(D)$
- Define $f(I)$ for finite width intervals and infinitesimal width intervals

Requires that f is monotone and differentiable.

Original program:

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observe(Normal(1.8,0.5),
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Original program:

```
observe(Normal(1.8,0.5),
        Interval(h,eps))
```

Answer: 1.75

Logarithmic ruler program:

```
observe(exp(Normal(1.8,0.5)),
        exp(Interval(h,eps)))
```

Answer: 1.75

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Requires that f is monotone and differentiable.

Original program:

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observe(Normal(1.8,0.5),
        Interval(h,eps))
```

Answer: 1.75

Logarithmic ruler program:

```
observe(exp(Normal(1.8,0.5)),
        exp(Interval(h,eps)))
```

Answer: 1.75

In general: `observe(f(D), f(I))` $\equiv$ `observe(D, I)`

$\implies$ **programs are invariant under parameter transformations**

Recap

- Paradoxical behaviour: seemingly equivalent probabilistic programs give different outputs
- Root of the problem: conditioning on measure-zero events is ambiguous
- Solution: condition on intervals
- Restores rejection sampling as ground truth semantics
- Model measure-zero events as a limit, computed using infinitesimal arithmetic
- Semantics of `observe(D, Interval(x, eps))` agrees with the old `observe(D, x)` in most cases
- Programs are now invariant under parameter transformations
- Implementation in Julia

Comments or questions?

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