Fast Coalgebraic Bisimilarity Minimization

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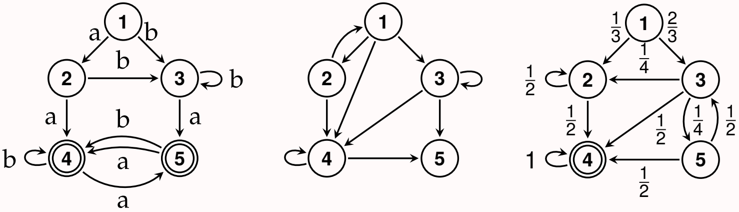
The Automaton Zoo

Deterministic finite automata, tree automata, (labeled) transition systems, weighted and probabilistic automata, Markov decision processes, ...



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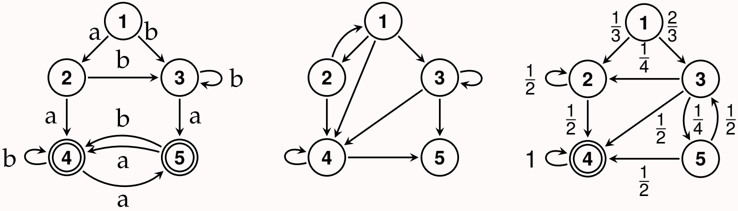

Automaton Minimization

Find and merge behaviorally equivalent states

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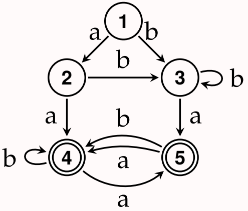

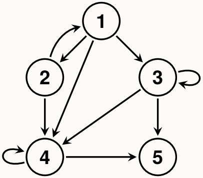

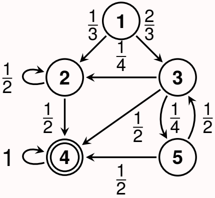

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Coalgebraic Bisimilarity Minimization

Algorithms that work for a general class of F-automata

Our contribution

a **fast** and **general** algorithm for minimizing automata

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- *General:* works for any computable coalgebra
- *Decent asymptotic complexity:* $O(\phi_F \cdot m \log n)$
- *Fast in practice:* no penalty for generality
- *Low memory usage:* important for large automata

Examples of Coalgebraic Automata

Automaton type	**Equivalence**	**Functor** $F(X)$
DFA	Language Equivalence	$2 \times A^X$
Transition Systems	Strong Bisimilarity	$\mathcal{P}(X)$
LTS	Strong Bisimilarity	$\mathcal{P}(A \times X)$
Weighted Systems	Weighted Bisimilarity	$M^{(X)}$
Markov Chain	Probabilistic Bisimilarity	$A \times \mathcal{D}(X)$
MDP	Probabilistic Bisimilarity	$\mathcal{P}(\mathcal{D}(X))$
Weighted Tree Automata	Backwards Bisimilarity	$M^{(\Sigma X)}$
Monotone Neigh. Frames	Monotone Bisimilarity	$\mathcal{N}(X)$
$\vdots$	$\vdots$	$\vdots$

Automaton types compose: $F \circ G, F + G, F \times G, \ldots$

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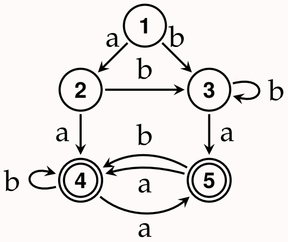
<td>
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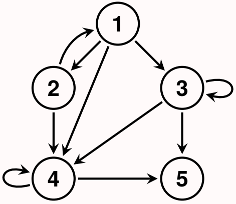
<td>
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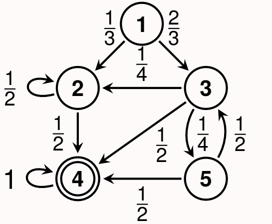
<td>
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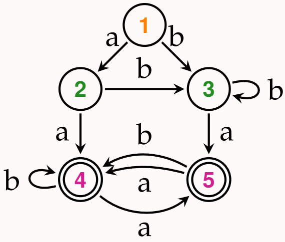
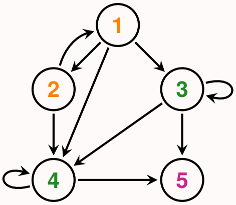
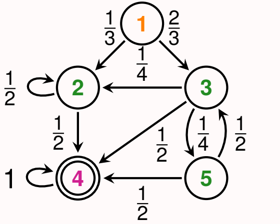
<tr><td>
</td><td>
</td><td>
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- a functor $F(X)$ – describes automaton type
- a coalgebra $t : C \to F(C)$ – the automaton

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The output:

- a partition $p : C \to C'$
  – the equivalence classes of bisimilar states
- s.t. $p(x) = p(y) \implies Fp(t(x)) = Fp(t(y))$
- $|C'|$ as small as possible

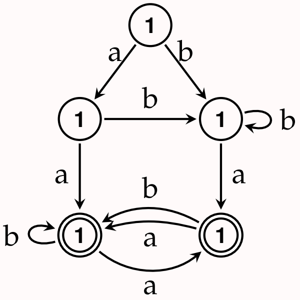
Sketch of our algorithm

- Assume all states are equivalent
- Pick an equivalence class
- Split equivalence class by *signature* (*normalised* outgoing transitions)
- Iterate until convergence

Key points

- Only recompute signatures of *changed* states
- Do not loop over *unchanged* states

Our algorithm: Minimizing a DFA



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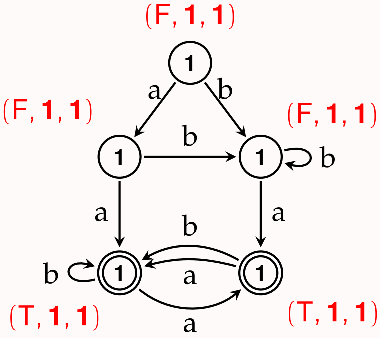

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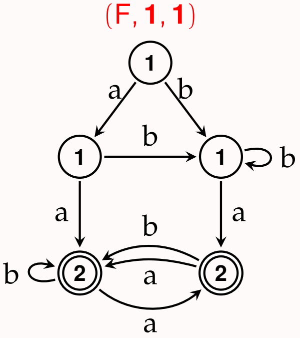

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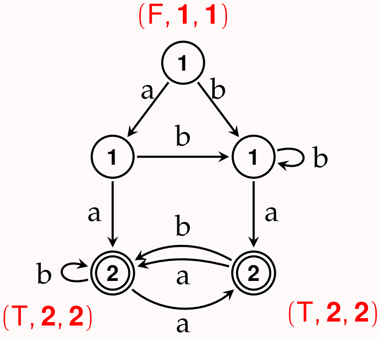

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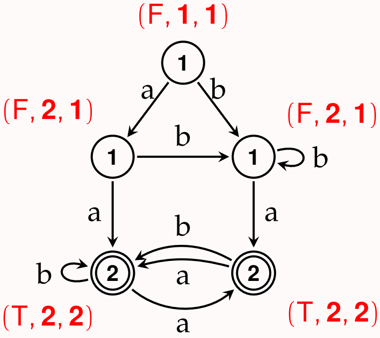

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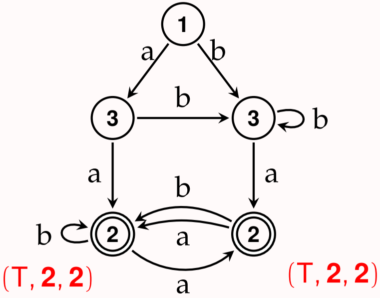

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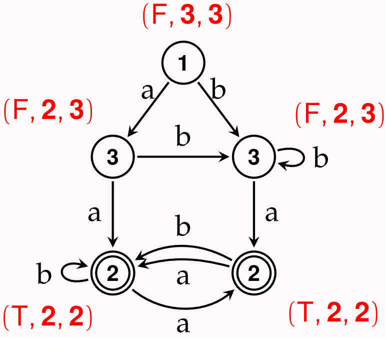

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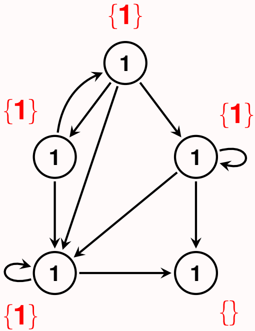

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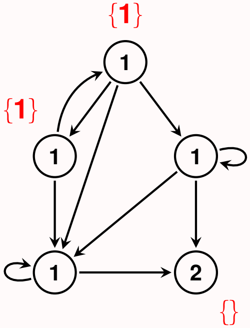

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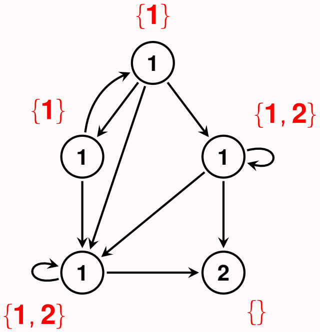

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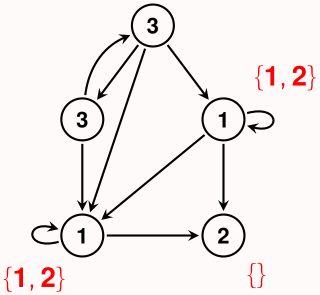

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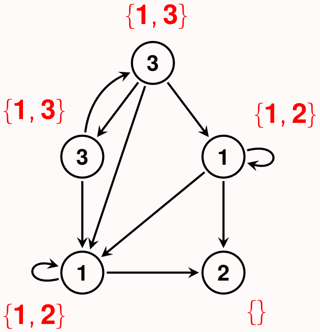

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Caveat

The previous examples are over-simplified, but the real algorithm is not complicated.

- The only complex part is *not looping over the unchanged states*
- See our paper for details

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- Ability to (re)compute signatures
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  - Total complexity: usually $O(km \log n)$
- What about the complexity of bookkeeping?
  - See paper for n-way partition refinement data structure
  - This is the only complex part of the algorithm

Comparison

	CoPaR	*DCPR*	*mCRL2*	*Boa*
Complexity	$O(m \log n)$	$O(\phi_F n^2)$	$O(m \log n)$	$O(\phi_F m \log n)$
Generality	Zippable	Coalg	LTS+	Coalg
Language	Haskell	Haskell	C++	Rust

benchmark		time (s)			memory (MB)	
type	**n**	***CoPaR***	***DCPR***	***Boa***	***DCPR***	***Boa***
fms	**1639440**	232	84	1.12	514×32	196
	4459455	–	406	4.47	1690×32	582
wlan	**607727**	105	855	0.28	147×32	42
	1632799	–	2960	0.79	379×32	93
wta(W)	**152107**	566	79	0.74	642×32	83
	944250	–	675	11.96	6786×32	1228
wta(Z)	**156913**	438	82	0.48	677×32	92
	1007990	–	645	16.75	5644×32	1325
wta(2)	**154863**	449	160	0.81	621×32	79
	1300000	–	1377	23.35	7092×32	1647

What is the cost of generality?

benchmark		time (s)		memory (MB)	
type	**n**	***mCRL2***	***Boa***	***mCRL2***	***Boa***
cwi	**2416632**	13.9	1.4	1780	249
	7838608	214.2	15.8	5777	814
	33949609	282.2	31.5	16615	2776
vasy	**6020550**	33.8	3.1	2124	520
	11026932	51.6	6.1	2768	619
	12323703	56.9	7.0	3103	734

For ***mCRL2***, we pick its best algorithm and self-reported time.
For ***Boa***, we report wall-clock time.

Conclusion

Minimization can be **generic** and **fast**

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Future

Other notions of equivalence (*e.g.*, branching).
Specialization by monomorphisation.
Integration into Storm (with Sebastian Junges).

(P.S., I'm looking for a postdoc position)